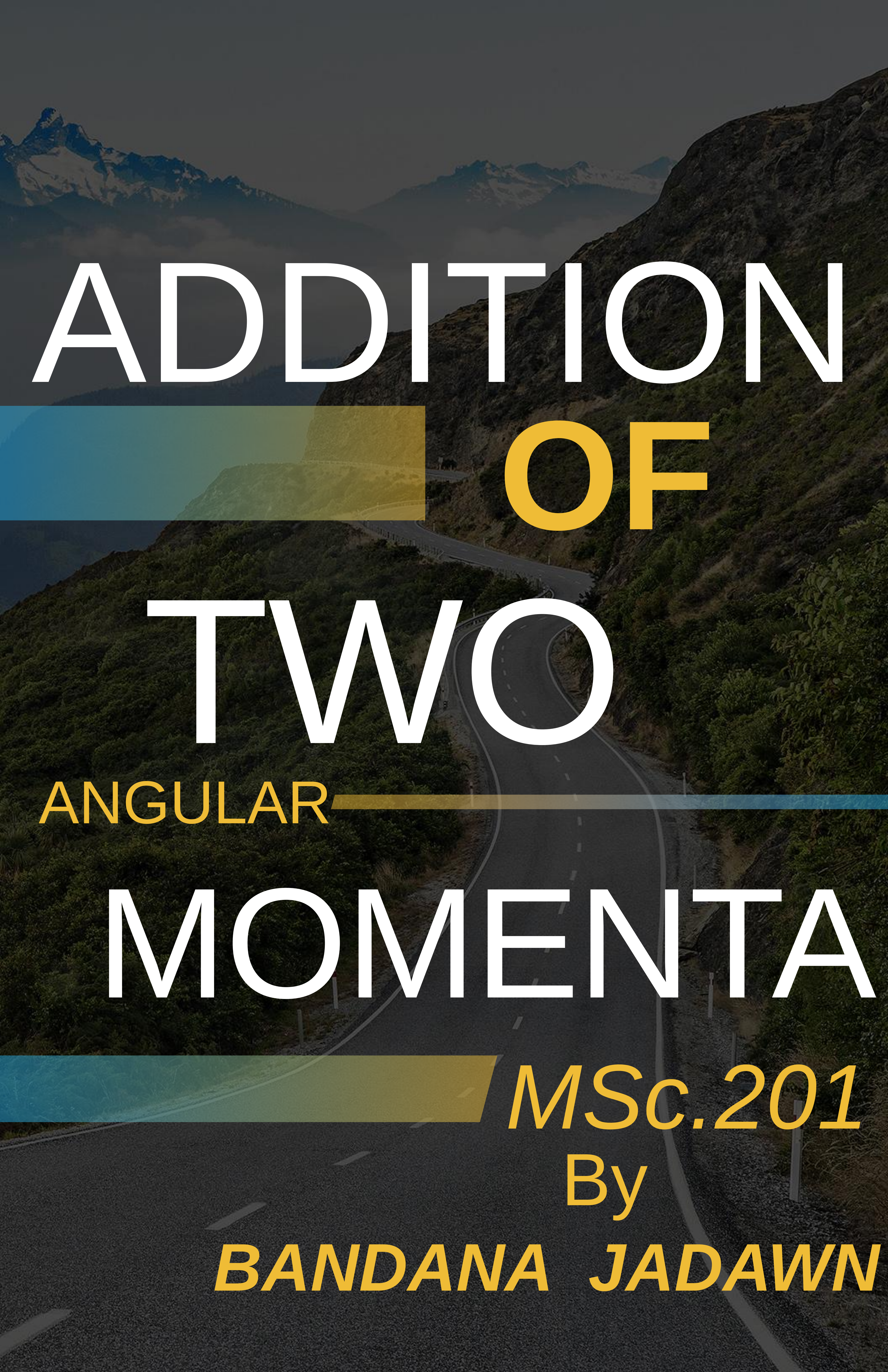

ADDITION OF TWO ANGULAR MOMENTA

MSc.201

By

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ADDITION OF TWO ANGULAR MOMENTA

Consider two angular momenta $\vec{J}_1$ and $\vec{J}_2$ which belongs to different subspaces 1 and 2.
The components $\vec{J}_1$ and $\vec{J}_2$ satisfy the usual Commutation relations of angular momentum.

$$[\hat{J}_{1x}, \hat{J}_{1y}] = i\hbar \hat{J}_{1z}, \quad [\hat{J}_{1y}, \hat{J}_{1z}] = i\hbar \hat{J}_{1x} \quad [\hat{J}_{1z}, \hat{J}_{1x}] = i\hbar \hat{J}_{1y} \quad -(1)$$

$$[\hat{J}_{2x}, \hat{J}_{2y}] = i\hbar \hat{J}_{2z} \quad [\hat{J}_{2y}, \hat{J}_{2z}] = i\hbar \hat{J}_{2x} \quad [\hat{J}_{2z}, \hat{J}_{2x}] = i\hbar \hat{J}_{2y} \quad -(2)$$

$\because$ J_1 and J_2 belongs to different spaces, their components commute -

$$[\hat{J}_{1j}, \hat{J}_{2k}] = 0 \quad \cdots \quad (j, k = x, y, z)$$

$\because$ J^2 and J_z commutes, hence they can have simultaneous eigen state.

Let $|j_1, m_1\rangle$ be the simultaneous eigen function of $\hat{J}_1^2$ and $\hat{J}_{1z}$ and $|j_2, m_2\rangle$ be the simultaneous eigen function of $\hat{J}_2^2$ and $\hat{J}_{2z}$.

$$\left.\begin{aligned}
\hat{J}_1^2 |j_1, m_1\rangle &= j_1(j_1+1)\hbar^2 |j_1 m_1\rangle \\
\hat{J}_{1z} |j_1 m_1\rangle &= m_1\hbar |j_1 m_1\rangle \\
\hat{J}_2^2 |j_2, m_2\rangle &= j_2(j_2+1)\hbar^2 |j_2 m_2\rangle \\
\hat{J}_{2z} |j_2 m_2\rangle &= m_2\hbar |j_2 m_2\rangle
\end{aligned}\right\} \quad (3)$$

$(2j_1+1)$ and $(2j_2+1)$ are the dimensions of the space of $\vec{J}_1$ and $\vec{J}_2$ respectively.
Therfore, the dimensions of the matrices of $\hat{J}_1^2$ & $\hat{J}_{1z}$ within the basis $\{|j_1 m_1\rangle\}$ and operators $\hat{J}_2^2$ & $\hat{J}_{2z}$ within the basis $\{|j_2 m_2\rangle\}$ are $(2j_1+1) \times (2j_1+1)$ & $(2j_2+1) \times (2j_2+1)$ respectively.

$\because$ These four operators $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2$ & $\hat{J}_{2z}$ commutes with each other, hence they can have diagonalized jointly by the same states.

Let $|j_1 j_2; m_1, m_2\rangle$ be the joint eigen states.

$$|j_1, j_2; m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle \quad -(4)$$

Since, the coordinates of $\vec{J}_1$ and $\vec{J}_2$ are independent therefore, eq (4) can be written as-

$$\left.\begin{aligned}
\hat{J}_1^2 |j_1, j_2; m_1, m_2\rangle &= j_1(j_1+1)\hbar^2 |j_1 j_2; m_1 m_2\rangle \\
\hat{J}_{1z} |j_1 j_2; m_1 m_2\rangle &= m_1\hbar |j_1 j_2; m_1 m_2\rangle \\
\hat{J}_2^2 |j_1 j_2; m_1 m_2\rangle &= j_2(j_2+1)\hbar^2 |j_1 j_2; m_1 m_2\rangle \\
\hat{J}_{2z} |j_1 j_2; m_1 m_2\rangle &= m_2\hbar |j_1 j_2; m_1 m_2\rangle
\end{aligned}\right\} \quad (5)$$

$\because$ we know $|j_1 j_2; m_1 m_2\rangle$ form a complete and orthonormal basis

Hence,

$$\sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2| = \left(\sum_{m_1} |j_1 m_1\rangle\langle j_1 m_1|\right)\left(\sum_{m_2} |j_2 m_2\rangle\langle j_2 m_2|\right) \quad — (6)$$

$\because$ $|j_1 m_1\rangle$ & $|j_2 m_2\rangle$ are complete

$$\sum_{m_1} |j_1 m_1\rangle\langle j_1 m_1| = 1 \quad \&$$

$$\sum_{m_2} |j_2 m_2\rangle\langle j_2 m_2| = 1$$

And since these states are also orthonormal, therefore

$$\left.\begin{aligned}
&\langle j_1' m_1' | j_1 m_1\rangle = \delta_{j_1' j_1}\delta_{m_1' m_1} \\
\text{similarly } &\langle j_2' m_2' | j_2 m_2\rangle = \delta_{j_2' j_2}\delta_{m_2' m_2}
\end{aligned}\right\}$$

$$\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} |j_1, j_2; m_1, m_2\rangle\langle j_1 j_2; m_1, m_2| = 1 \quad — (7)$$

and orthonormal

$$\langle j_1', j_2'; m_1' m_2' | j_1 j_2; m_1 m_2\rangle = \langle j_1' m_1' | j_1 m_1\rangle\langle j_2' m_2' | j_2 m_2\rangle$$
$$= \delta_{j_1' j_1}\delta_{j_2' j_2}\delta_{m_1' m_1}\delta_{m_2' m_2} \quad — (8)$$

Now, we have to introduce two operators

$$\hat{J}_{1\pm} = \hat{J}_{1x} \pm i\hat{J}_{1y}$$
$$\& \quad \hat{J}_{2\pm} = \hat{J}_{2x} \pm i\hat{J}_{2y}.$$

Now the action eq of these operators on state $|j_1 j_2; m_1 m_2\rangle$ -

$$\left.\begin{aligned}
\hat{J}_{1\pm} |j_1 j_2; m_1 m_2\rangle &= \hbar\sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\, |j_1 j_2; m_1 \pm 1, m_2\rangle \\
\hat{J}_{2\pm} |j_1, j_2; m_1 m_2\rangle &= \hbar\sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\, |j_1 j_2; m_1, m_2 \pm 1\rangle
\end{aligned}\right\}$$

The addition of angular momenta consists of finding the eigen values and eigen vectors of $\hat{J}^2$ and $\hat{J}_z$ in terms of the eigen values and eigen vectors of $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2$ & $\hat{J}_{2z}$.

$$\hat{J} = \hat{J}_1 + \hat{J}_2 \quad -(10)$$

$\because$ Matrices of $\hat{J}_1$ & $\hat{J}_2$ have different dimensions in general.

$\therefore$ Addition of $\hat{\vec{J}}$ defined by eq (10) is not an addition of matrices.

By adding eq (1) & (2), we can easily shown that the components of $\vec{J}$ satisfy the commutation relations

$$[\hat{J}_x\ \hat{J}_y] = i\hbar\hat{J}_z \quad [J_y\ J_z] = i\hbar\hat{J}_x \quad [J_z\ J_x] = i\hbar\hat{J}_y \quad -(11)$$

$\because$ $\hat{\vec{J}}^2 = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + J_{1+}J_{2+} + J_{1-}J_{2-}$

which leads to

$$[\hat{\vec{J}}^2, \hat{\vec{J}}_1^2] = [\hat{\vec{J}}^2, \hat{\vec{J}}_2^2] = 0$$

and $[\hat{J}^2, \hat{J}_z] = [\hat{J}_1^2, \hat{J}_z] = [\hat{J}_2^2, \hat{J}_z] = 0$

but $\hat{J}_{1z}$ & $\hat{J}_{2z}$ do not commute seperately with $\hat{\vec{J}}^2$.

$$[\hat{J}^2, \hat{J}_{1z}] \neq 0, \quad (J^2, \hat{J}_{2z}] = 0$$

$\because$ $\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z$ form a complete set of commuting operators hence they can be diaginalized simultaneously

Let $|j_1 j_2; j, m\rangle$ be the simultaneous eigen state.

$$\left.\begin{aligned}
\hat{J}_1^2 |j_1 j_2; j, m\rangle &= j_1(j_1+1)\hbar^2 |j_1 j_2; j, m\rangle \\
\hat{J}_2^2 |j_1 j_2; j m\rangle &= j_2(j_2+1)\hbar^2 |j_1 j_2; j m\rangle \\
J^2 |j_1 j_2; j, m\rangle &= j(j+1)\hbar^2 |j_1 j_2; j, m\rangle \\
\hat{J}_z |j_1 j_2; j, m\rangle &= m\hbar |j_1 j_2; j, m\rangle
\end{aligned}\right\} \quad -(12)$$

For each value 'j' the number 'm' has $(2j+1)$ allowed values ($m = -j$ to $+j$).

$\because$ J_1 and J_2 are usually fixed, we will be using $|j, m\rangle$ to abbreviate $|j_1 j_2; j, m\rangle$ & the set of vectors $\{|j, m\rangle\}$ forms a complete and orthonormal basis.

CLEBSCH - GORDAN COEFFICIENTS

Since, we know the $|j,m\rangle$ is the state in which $\hat{J}^2$ & $\hat{J}_z$ have fixed values i.e. $j(j+1)$ and m. $|j_1, j_2; m_1, m_2\rangle$ is the state in which $\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}$ have fixed values.

The $\{|j_1, j_2; m_1, m_2\rangle\}$ and $\{|j,m\rangle\}$ bases can be connected by means of a transformation as follows.

By using completeness equation

$$|j,m\rangle = \left(\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1, j_2; m_1 m_2\rangle\langle j_1, j_2; m_1 m_2| \right) |j,m\rangle$$

$$|j,m\rangle = \sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j,m\rangle\, |j_1 j_2; m_1 m_2\rangle \quad \text{— (13)}$$

The coefficients $|j_1, j_2; m_1, m_2|j,m\rangle$ which depends only on the quantities j_1, j_2, j, m_1, m_2 and m, are the matrix elements of this transformation which connects the $\{|j,m\rangle\}$ and $\{|j_1, j_2; m_1, m_2\rangle\}$ bases. These coefficients are called the Clebsch - Gordan coefficients.

These coefficients are taken to be real by convention

$$\langle j_1, j_2; m_1, m_2 | j,m\rangle = \langle j,m | j_1 j_2; m_1, m_2\rangle \quad \text{—(14)}$$

The orthonormalization relation for the Clebsch-Gordan coefficients -

$$\sum_{m_1 m_2} \langle j', m' | j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2 | j, m\rangle = \delta_{j'j}\, \delta_{m'm} \quad \text{—(15)}$$

Since, these coefficients are real, this relation can be rewritten as-

$$\sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j', m'\rangle\langle j_1 j_2; m_1 m_2 | j, m\rangle = \delta_{j'j}\, \delta_{m'm} \quad \text{—(16)}$$

which leads to

$$\sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j, m\rangle^2 = 1 \quad \text{—(17)}$$

Similarly,

$$\sum_{j} \sum_{m=-j}^{j} \langle j_1 j_2; m_1' m_2' | j, m\rangle\langle j_1 j_2; m_1 m_2 | j, m\rangle = \delta_{m_1' m_1}\, \delta_{m_2' m_2} \quad \text{—(18)}$$

and in particular

$$\boxed{\sum_{j} \sum_{m} \langle j_1 j_2; m_1 m_2 | j, m\rangle^2 = 1} \quad \text{—— (19)}$$


THANK
YOU
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